Appendix

Conditions for diffusion instability

In order to examine the conditions for diffusion instability to arise (Okubo, 1980; Murray, 1989), the following predator-prey equations are used:

$$\frac{dN}{dt} = f(N)N - g(N,P)P + D_N \frac{\partial^2 N}{\partial x^2} \tag{1}$$

$$\frac{dP}{dt} = eg(N,P)P - \mu P + D_P \frac{\partial^2 P}{\partial x^2}, \tag{2}$$

which can be summarized in a more general form:

$$\frac{\partial N}{\partial t} = F(N,P) + D_N \frac{\partial^2 N}{\partial x^2} \tag{3}$$

$$\frac{\partial P}{\partial t} = G(N,P) + D_P \frac{\partial^2 P}{\partial x^2} \tag{4}$$

This system will display diffusion driven instability if there is a spatially uniform state where preys and predators coexist in a stable equilibrium that becomes unstable to certain spatially inhomogeneous small perturbations. This property can be outlined analytically by means of three conditions:

1. A feasible coexistence equilibrium point must exist. Thus, the system:

$$F(N^*, P^*) = 0 \tag{5}$$

$$G(N^*, P^*) = 0 \tag{6}$$

must have a feasible solution, $N^* > 0$ and $P^* > 0$.

2. The coexistence point must be stable when subjected to spatially homogeneous small perturbations from the spatially uniform state. To assess stability, the so-called community matrix (Levins, 1968) must be evaluated at the equilibrium point (N^*, P^*):

$$J^* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{11} = \left.\frac{\partial F}{\partial N}\right|_{(N^*,P^*)} \quad a_{12} = \left.\frac{\partial F}{\partial P}\right|_{(N^*,P^*)}$$

$$a_{21} = \left.\frac{\partial G}{\partial N}\right|_{(N^*,P^*)} \quad a_{22} = \left.\frac{\partial G}{\partial P}\right|_{(N^*,P^*)} \tag{7}$$

If the trace of J^* is negative while its determinant is positive the equilibrium point will be stable in front of spatially homogeneous small perturbations:

$$a_{11} + a_{22} < 0 \tag{8}$$

$$a_{11}a_{22} - a_{21}a_{12} > 0 \tag{9}$$

3. The coexistence stable point (N^*, P^*) must be unstable when subjected to inhomogeneous small perturbations. The condition for that to occur depends also on the elements of the community matrix, and on the relative diffusion, $d = D_P/D_N$. It can be expressed as (Murray, 1989):

$$d\, a_{11} + a_{22} > 2\sqrt{d}\sqrt{det(J^*)} \tag{10}$$

The fulfillment of the three conditions ensures the generation of spatial pattern through diffusion driven instability.

Prey-dependent models: absence of Turing structures

In prey dependent models, the predator functional response, $g(N)$, does not depend on predator abundance. Therefore, the general equations (1)-(2) become:

$$\frac{\partial N}{\partial t} = f(N)N - g(N)P + D_N \frac{\partial^2 N}{\partial x^2} \tag{11}$$

$$\frac{\partial P}{\partial t} = eg(N)P - \mu P + D_P \frac{\partial^2 P}{\partial x^2} \tag{12}$$

It has already been shown in previous studies (Segel and Jackson, 1972) that the system (11)-(12) cannot present diffusion instability. The reason can be easily understood if the entries of the community matrix are evaluated at the equilibrium point, e.g., $N^* = g^{-1}(\mu/e)$, $P^* = e/\mu \, N^* f(N^*)$:

$$a_{11} = f(N^*) + N^* \left.\frac{df}{dN}\right|_{N^*} - \left.\frac{dg}{dN}\right|_{N^*} P^* \quad (13)$$

$$a_{12} = -g(N^*) = -\frac{\mu}{e} \quad (14)$$

$$a_{21} = e \left.\frac{dg}{dN}\right|_{N^*} P^* \quad (15)$$

$$a_{22} = eg(N^*) - \mu = 0 \quad (16)$$

From eq (6), it can be seen that a_{22} must be null. Thus, stability condition (8) will become $a_{11} < 0$, while condition needed for diffusion driven instability to occur (10) will become $a_{11} > \frac{2\sqrt{d}}{d}\sqrt{det(J^*)} > 0$. Therefore, the two conditions cannot be fulfilled simultaneously.

References

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